Section 3.2: Solving Systems of Linear Equations Using Matrices

As you may recall from College Algebra or Section 1.3, you can solve a system of linear equations in two variables easily by applying the substitution or addition method. Since these methods become tedious when solving a large system of equations, a suitable technique for solving such systems of linear equations will consist of Row Operations. The sequence of operations on a system of linear equations are referred to equivalent systems, which have the same solution set.

Row Operations

1. Interchange any two rows.

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 5 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 3 \end{bmatrix}$$

2. Replace any row by a nonzero constant multiple of itself.

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 8 \end{bmatrix} \quad \frac{1}{4}R_2 \to R_2 \quad \begin{bmatrix} 2 & -1 & 3 \\ 1 & -\frac{1}{2} & 2 \end{bmatrix}$$

3. Replace any row by the sum of that row and a constant multiple of any other row.

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right] \quad -2R_1 + R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array}\right]$$

Row Reduced Form

An m x n augmented matrix is in row-reduced form if it satisfies the following conditions:

1. Each row consisting entirely of zeros lies below any other row having nonzero entries.

$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{array}\right] \quad \text{the correct row-reduced form} \quad \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

2. The first nonzero entry in each row is 1 (called a leading 1).

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right] \quad \text{the correct row-reduced form} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

3. If a column contains a leading 1, then the other entries in that column are zeros.

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \text{ the correct row-reduced form } \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

4. In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.

$$\left[\begin{array}{cc|c} 0 & 1 & -2 \\ 1 & 0 & 3 \end{array}\right] \text{ the correct row-reduced form } \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

Example 1: Determine which of the following matrices are in row-reduced form. If a matrix is not in row-reduced form, state which condition is violated.

a. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$

d. $\left(\begin{array}{ccc|c} 1 & -9 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

b. $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

e. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 2 & 6 \end{array}\right)$

c. $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \end{array}\right)$

f. $\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 5 \end{array}\right)$

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The Gauss-Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.

2. Use row operations to write the augmented matrix in row reduced form. If at any point a row in the matrix contains zeros to the left of the vertical line and a nonzero number to its right, stop the process, as the problem has no solution.

3. Read off the solution(s).

There are three types of possibilities after doing this process.

Unique Solution

Example 2: The following augmented matrix in row-reduced form is equivalent to the augmented matrix of a certain system of linear equations. Use this result to solve the system of equations.

$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Example 3: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$x + 2y = 1$$
$$2x + 3y = -1$$

Example 4: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{aligned} 3x + y &= 1 \\ -7x - 2y &= -1 \end{aligned}$$

Example 5: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{aligned} y - 8z &= 9 \\ x - 2y + 3z &= -3 \\ 7y - 5z &= 12 \end{aligned}$$

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Example 6: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{aligned} 2x + 4y - 6z &= 38 \\ x + 2y + 3z &= 7 \\ 3x - 4y + 4z &= -19 \end{aligned}$$

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Infinite Number of Solutions

Example 7: The following augmented matrix in row-reduced form is equivalent to the augmented matrix of a certain system of linear equations. Use this result to solve the system of equations.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Example 8: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned}$$

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A System of Equations That Has No Solution

In using the Gauss-Jordan elimination method the following equivalent matrix was obtained (note this matrix is not in row-reduced form, let's see why):

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Look at the last row. It reads: 0x + 0y + 0z = -1, in other words, 0 = -1!!! This is never true. So the system is inconsistent and has no solution.

Systems with No Solution

If there is a row in the augmented matrix containing all zeros to the left of the vertical line and a nonzero entry to the right of the line, then the system of equations has no solution.

Example 9: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{aligned} 2x + 3y &= 2 \\ x + 3y &= -2 \\ x - y &= 3 \end{aligned}$$

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Example 10: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{aligned} -x + 3y - 4z &= 12 \\ 4x - 12y + 16z &= -36 \end{aligned}$$

Example 11: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{aligned} 3x + y - 4z &= 6 \\ -15x - 5y + 20z &= -36 \end{aligned}$$

Example 12: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{aligned} 2x - 3y &= 13 \\ x + y &= -1 \\ x - 4y &= 14 \end{aligned}$$